A Model of Type Theory in Cubical Sets

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Abstract

We present a model of type theory with dependent product, sum, and identity, in cubical sets. We describe a universe and explain how to transform an equivalence between two types into an equality. We also explain how to model propositional truncation and the circle. While not expressed internally in type theory, the model is expressed in a constructive metalogic. Thus it is a step towards a computational interpretation of Voevodsky's Univalence Axiom.

1998 ACM Subject Classification F.4.1 Mathematical Logic, F.3.2 Semantics of Programming Languages

Keywords and phrases models of dependent type theory, cubical sets, univalent foundations

Digital Object Identifier 10.4230/LIPIcs.TYPES.2013.107

La théorie singulière classique utilise des simplexes*; dans la suite de ce chapitre, nous aurons besoin d'une définition équivalente, mais utilisant des* cubes*; il est en effet évident que ces derniers se prêtent mieux que les simplexes à l'étude des produits directs, et, a fortiori, des espaces fibrés qui en sont la généralisation.*

(J. P. Serre, Thèse, Paris,1951 [21])

1 Introduction

In [16], Voevodsky proposes a new axiom in dependent type theory: the Univalence Axiom. This opens up for many improvements for the encoding of mathematics in type theory in general: function extensionality, identification of isomorphic structures, etc.

In order to preserve the good computational properties of type theory it is crucial that postulated constants have a computational interpretation. Concerning univalence, this is an important open problem. One way of attacking this problem is by constructing a model of the new axiom, in type theory itself, or at least in a constructive metalogic. The computational interpretation can then be obtained through the semantics, for example, by evaluating a term of type N (natural numbers) in the model.

The model of type theory with the Univalence Axiom given by Voevodsky [16] is based on Kan simplicial sets. A problem with a constructive approach to Kan simplicial sets is that degeneracy is in general undecidable [3]. This problem makes it impossible to use the Kan simplicial set model as it is to obtain a computational interpretation of univalence.

We present a model of dependent type theory in cubical sets. This can be seen as a generalization of Bishop's notion of *set* [4]. While not expressed internally in type theory,

19th International Conference on Types for Proofs and Programs (TYPES 2013).
Editors: Ralph Matthes and Aleksy Schubert; pp. 107–128
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

this model is expressed in a constructive metalogic. It can be seen as a simplification and a constructive version of the Kan simplicial set model of type theory [16, 1].

The first combinatorial description of homotopy groups by Kan used cubical sets [15]; see [7], [27] for a more recent account. Our presentation of cubical sets amounts to have a formal representation of cubes seen as continuous maps $[0,1]^I \to X$, where I is a finite set of symbols, instead of using only continuous maps $[0,1]^n \to X$. If $I = x_1, \ldots, x_n$ such a continuous map u can be seen as a function of $x_1, \ldots, x_n$ which vary in the unit interval. We can then consider for instance $u(x_i = 0)$, which is the quantity u where we set x_i to be 0, or we can introduce a new symbol y and consider u to be a quantity as a function of $x_1, \ldots, x_n, y$, which is actually independent of y. We formalize this by defining a cubical set to be a covariant presheaf on a suitable base category, where objects are finite sets of symbols and maps are substitution. This opens connections with the theory of nominal sets [20, 19].

Following e.g. [11], we can give a model of type theory where a context is interpreted by a cubical set. Like for the classical model based on simplicial sets where one restricts the model to Kan fibrations, we restrict our model by requiring a certain *Kan structure* on dependent types. Like in Kan's original paper [15], such a Kan structure requires fillers of open boxes. However, in order for this structure to be preserved–in a constructive metalogic–under all type forming operations, in particular Π, a certain uniformity condition is required on the choice of the fillers. This structure is essential for validating the elimination rule of identity types.

The strengthening of the Kan condition is natural given the reformulation of the notion of cubical sets that we present in the first section, and the connection mentioned above with nominal sets.

In this paper we present the semantics of dependent products, sums and identity types. We also show how to interpret the universe, but only sketch one special case how one could define the Kan structure on the universe. We also only describe how to transform an equivalence between two small types into a path between these types. Based on the model described in the first version of this paper (a nominal version of it) C. Cohen, A. Mörtberg and the last two authors have implemented a type checker[1]. This implementation supports computing with the Univalence Axiom and Kan operations for the universe.

The paper is structured as follows. In the next two sections we introduce the category of names and substitutions and we define cubical sets. In Section 4 we explain the presheaf semantics of type theory in the special case of cubical sets. In the next two sections we define the uniform Kan condition and we give examples of cubical sets. In Section 7 we show that Kan cubical sets are a model for dependent types. In the last section we show how identity types can be interpreted in the Kan cubical set model, and describe the universe as a cubical set (and only indicate how Kan fillings can be given), and how to transform an equivalence into an equality of types. Finally, we explain how to represent in our model spaces up to homotopy such as the sphere, and the operation of propositional truncation, giving in particular a new computational interpretation of the axiom of description [26, Introduction].

2 The category of names and substitutions

We start by fixing a countable discrete set of names or symbols, hereafter called the *name space*, such that 0 and 1 are not names.

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[1] Available at: `http://github.com/simhu/cubical`

▶ **Definition 1.** The category $\mathcal{C}$ of names and substitutions has as objects all finite decidable subsets of the name space, denoted by $I, J, K, \ldots$. A morphism $f : I \to J$ is a map $I \to J \cup \{0, 1\}$ such that $f(i) = f(j)$ iff $i = j$ whenever $f(i)$ and $f(j)$ are in J.

Notice that $\{0, 1\}$ is disjoint from J since J is a set of names. We say that i is in the *domain* of f, or that $f(i)$ is *defined*, if $f(i)$ is an element of J.[2] So the condition for f being a morphism can be reformulated by saying that f is injective on its domain.

Clearly, $1_I : I \to I$ defined by $1_I(i) = i$ for all $i \in I$ is a morphism. If $f : I \to J$ and $g : J \to K$ are morphisms, we define the composition $g \circ f$ by $(g \circ f)(i) = g(f(i))$ if i is in the domain of f, and $(g \circ f)(i) = f(i)$ if $f(i) = 0, 1$. Clearly, $g \circ f : I \to K$ is a morphism. We shall write fg for the composition $g \circ f$, so first f and then g. It is not difficult to see that composition is associative and that $1_I f = f = f 1_J$. Hence $\mathcal{C}$ is a category. From now on, we may simply write 1 instead of 1_I.

Every $f : I \to J$ has a unique extension to a map $I \cup \{0, 1\} \to J \cup \{0, 1\}$ that is the identity on $\{0, 1\}$, and this canonical extension respects composition. Together with $I \mapsto I \cup \{0, 1\}$ we get a functor $\mathcal{C} \to \mathsf{Set}$.

We think of $f : I \to J$ as a substitution with renaming, where the only values we can substitute are 0 and 1. In particular we have for any x in I two substitutions $(x = b) : I \to I - x$, for $b = 0, 1$, defined by $(x = b)(y) = y$ if $y \neq x$ and $(x = b)(x) = b$. These are the *face maps*. Thus there are $2n$ face maps when I has n elements, that is, in dimension n (where simplicial sets have $n + 1$ face maps).

We say that a map $f : I \to J$ is a *degeneracy map* iff all elements in I are in the domain of f. For instance, if $I \subseteq J$ the canonical inclusion $I \to J$ defines a degeneracy map. If x is not in I the inclusion map $I \to I, x$ will be written as ι_x. We have two face maps $(x = 0), (x = 1) : I, x \to I$ and we have $\iota_x(x = 0) = \iota_x(x = 1) = 1_I$, which is one example of a *cubical identity*. There are many more cubical identities, often implicit in the notations. We also have the following result (cf. simplicial sets): every morphism f has a unique decomposition $f = gh$ where g is a composite of face maps and h is a degeneracy map.

If $f : I \to J$ is defined on x, we write $f - x : I - x \to J - f(x)$ for the map defined by $(f - x)(y) = f(y)$ if y is in $I - x$.

If $f : I \to J$ and x is not in I and y is not in J, we can extend f to a map $(f, x = y) : I, x \to J, y$ by sending x to y.

3 Cubical sets

▶ **Definition 2.** A *cubical set* is a covariant functor $\mathcal{C} \to \mathsf{Set}$.

Let X be a cubical set. Then we have sets $X(I)$ and set maps (called *restrictions*) $X(I) \to X(J)$, $u \longmapsto uf$ for any morphism $f : I \to J$, such that $u1 = u$ and $u(fg) = (uf)g$. Another notation for uf would be $X(f)(u)$.

A cubical set X is a presheaf on the category $\mathcal{C}^{op}$. Any finite set of directions I represents by the Yoneda embedding $\mathbf{y} : \mathcal{C}^{op} \to \mathsf{Set}^{\mathcal{C}}$ a cubical set $\mathbf{y}I$, which can be thought of as a formal representation of $[0, 1]^I$. An element of $X(I)$ can then be seen as a formal representation of a "continuous" map $[0, 1]^I \to X$, and it is natural to call an element of $X(I)$ an I-cube.

[2] In a previous attempt, we have been considering the category of finite sets with maps $I \to J + 2$ (i.e. the Kleisli category for the monad $I + 2$). This category appears on pages 47–48 in Pursuing Stacks [10] as "in a sense, the smallest test category".

For finite sets of names we will write commas instead of unions and often omit curly braces; e.g. we write I, x for $I \cup \{x\}$, $I - x$ for $I - \{x\}$, and $X(x_1, \ldots, x_n)$ for $X(\{x_1, \ldots, x_n\})$.

We think of u in $X(I)$ as meaning that u may depend on the names in I, and only on those names; we think of uf in $X(J)$ as the element we obtain by performing the substitution f on u, possibly combined with renaming and/or adding variables. An element of $X()$ represents a point, an element ω of $X(x)$ a line connecting the points $\omega(x = 0)$ and $\omega(x = 1)$ in $X()$. An element in $X(x, y)$ represents a square. We then follow some notations similar to the ones in first-order logic by writing $u = u(x_1, \ldots, x_n)$ when u is in $X(x_1, \ldots, x_n)$. This is similar to saying that u may depend at most on the names $x_1, \ldots, x_n$. In doing so we always implicitly assume that the names $x_1, \ldots, x_n$ are pairwise distinct; the order of the names in $X(x_1, \ldots, x_n)$ does not matter. Applying a face map will now be expressed by actually performing the substitution. For example, we have that $u(x = 0)$ is in $X(y)$ whenever u is in $X(x, y)$:

$$
\begin{tikzcd}
u(0,1) \arrow[r, "u(x,1)"] & u(1,1) \\
u(0,0) \arrow[u, "u(0,y)"] \arrow[r, "u(x,0)"'] \arrow[ur, phantom, "u(x,y)"] & u(1,0) \arrow[u, "u(1,y)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
u(0,0) \arrow[r, "u(x,0)"] & u(1,0) \\
u(0,0) \arrow[u, "u(0,0)"] \arrow[r, "u(x,0)"'] \arrow[ur, phantom, "u(x,0)"] & u(1,0) \arrow[u, "u(1,0)"']
\end{tikzcd}
$$

If v is an $I - x$ cube of X then we can consider $v\iota_x$ which is an I-cube of X (we recall that $\iota_x : I - x \to I$ is the canonical inclusion). The map $v \longmapsto v\iota_x$ is injective (we have $v\iota_x(x = 0) = v$) and it is natural to identify v and $v\iota_x$, thus considering $X(I - x)$ to be a subset of $X(I)$. An example is the degenerate right square above.

If u is in $X(I)$ and x is in I, there may exist a v in $X(I - x)$ such that $u = v\iota_x = v$. Intuitively, this means that x "does not occur" in u, or that u is "independent" of x. One sometimes uses the notation $x\#u$ to express this relation. In general, this relation does not need to be decidable.

If X is a cubical set and a and u are two points ($\emptyset$-cube) of X we can define a new cubical set $\mathsf{Id}_X\ a\ u$ by taking an element in $(\mathsf{Id}_X\ a\ u)(I)$ to be an I, x-cube ω of X where x is a fresh variable (i.e. $x \notin I$), such that $\omega(x = 0) = a$ and $\omega(x = 1) = u$. The name x is "bound" in this operation so that another I, x'-cube ω' is equal to ω iff $\omega'(x' = x) = \omega$. We introduce a new binding operation $\langle x \rangle \omega$ which defines this I-cube of $\mathsf{Id}_X\ a\ u$. One way to make this notion precise is to assume a choice function on the set of names which selects a fresh name for any finite subset and define $\langle x \rangle \omega$ to be $\omega(x = x_I)$ where x_I is the fresh name given by the choice function. (This is the solution suggested in [22].)

The corresponding category with the same objects and morphisms $I \to J \cup \{0\}$ has been already considered as the category of *partial injections*. It has been shown by Staton that the category of covariant presheaves over this category is equivalent to the category of nominal sets with one restriction operation (see [20, exercise 9.7]). Using the same method, we can associate in a canonical way a nominal set to any cubical set. A category equivalent to to the category of cubical sets is presented in [19].

4 Cubical sets as a presheaf model

We will now recall how cubical sets, as does any presheaf category, give rise to a model of dependent type theory. We use Dybjer's notion of *category with families* (CwF) to devise such a model [9, 8, 11]. We stress the fact that such a structure is described by a generalized algebraic theory [5]. To give a CwF is to give:

1. interpretations (as sets) for the sorts of contexts, context morphisms (substitutions), types and terms;
2. operations;
3. checking equations.

This amounts to validate the rules given in Figure 1. Note that we use polymorphic notation to increase readability as in [5, 9]; e.g. without this convention we should have written $\mathsf{p}_{\Gamma,A}$ for the first projection $\mathsf{p}\colon \Gamma.A \to \Gamma$. Also, we leave the type parameters implicit, e.g. $(A\sigma)\delta = A(\sigma\delta)$ tacitly assumes the premises $\sigma\colon \Delta \to \Gamma$, $\delta\colon \Theta \to \Delta$ and $\Gamma \vdash A$. These points are also stressed in [25, Sec. 1] and [9].

We will now describe how cubical sets give rise to such a structure. This construction works for any presheaf category and is described in [11, Sec. 4]. Instead of using contravariant presheaves, we use covariant presheaves and write composition in diagram order.

A context Γ, written $\Gamma \vdash$, is interpreted by a cubical set, and context morphisms $\sigma : \Delta \to \Gamma$ are interpreted as cubical set maps (i.e. natural transformations), that is we have $(\sigma\beta)f = \sigma(\beta f)$ if β is a I-cube of Δ. A dependent type $\Gamma \vdash A$ is given by sets $A\alpha$ for each I-cube α of Γ together with maps (also called *restrictions*) $A\alpha \to A\alpha f$, $u \longmapsto uf$ for each $f\colon I \to J$, satisfying $u1 = u$ and $u(fg) = (uf)g$. Another way to express this is to say that A is a covariant presheaf on the category of elements of Γ, where the category of elements of Γ is given by objects (I, α) with $\alpha \in \Gamma(I)$, and morphisms $f\colon (I, \alpha) \to (J, \beta)$ given by $f\colon I \to J$ in $\mathcal{C}$ such that $\beta = \alpha f$. A section (or term) $\Gamma \vdash a : A$ is defined by giving an element $a\alpha$ in $A\alpha$ for each I-cube α of Γ in such a way that $(a\alpha)f = a(\alpha f)$ for any $f\colon I \to J$. The empty context $()$ is given by the cubical set with exactly one I-cube for each I. Given $\Gamma \vdash A$ and $\sigma\colon \Delta \to \Gamma$ we define $\Delta \vdash A\sigma$ by $(A\sigma)\alpha = A(\sigma\alpha)$ and the induced maps; likewise, substitution in a term $\Gamma \vdash a : A$ is given by $(a\sigma)\alpha = a(\sigma\alpha)$. If $\Gamma \vdash A$, we define the cubical set $\Gamma.A$ by taking as I-cubes of $\Gamma.A$ pairs (α, u) with α an I-cube of Γ and u in $A\alpha$. For $f : I \to J$ we define $(\alpha, u)f = (\alpha f, uf)$. The first projection $\mathsf{p}\colon \Gamma.A \to \Gamma$, $\mathsf{p}(\alpha, u) = \alpha$ becomes thus a context morphism, and the second projection $\mathsf{q}(\alpha, u) = u$ a section $\Gamma.A \vdash \mathsf{q} : A\mathsf{p}$ corresponding to the first de Bruijn index. For $\Gamma \vdash A$, $\sigma\colon \Delta \to \Gamma$ and $\Delta \vdash u : A\sigma$ we give $(\sigma, u)\colon \Delta \to \Gamma.A$ by $(\sigma, u)\beta = (\sigma\beta, u\beta)$. This concludes the description of the CwF without type formers.

We now describe how to interpret Π and Σ. If $\Gamma \vdash A$ and $\Gamma.A \vdash B$, we define the type $\Gamma \vdash \Pi\, A\, B$ as follows. For each I-cube α of Γ, an element w of $(\Pi\, A\, B)\alpha$ is a family (w_f) indexed by $f : I \to J$ such that

$$w_f \in \prod_{u \in A\alpha f} B(\alpha f, u)$$

is a dependent function and $(w_f(u))g = w_{fg}(ug)$ for $g\colon J \to K$ and $u \in A\alpha f$. We define the family wf in $(\Pi\, A\, B)\alpha f$ by putting $(wf)_g = w_{fg}$, which completes the definition of $\Gamma \vdash \Pi\, A\, B$. Given $\Gamma.A \vdash b : B$ we interpret $\Gamma \vdash \lambda\, b : \Pi\, A\, B$ by $((\lambda\, b)\alpha)_f(u) = b(\alpha f, u)$ for u in $A\alpha f$. Application $\Gamma \vdash \mathsf{app}(w, u) : B[u]$ (where $[u] = (1, u)\colon \Gamma \to \Gamma.A$) of $\Gamma \vdash w : \Pi\, A\, B$ to $\Gamma \vdash u : A$ is given by $\mathsf{app}(w, u)\alpha = (w\alpha)_1(u\alpha)$. We get $\mathsf{app}((\lambda\, b), u)\alpha = ((\lambda\, b)\alpha)_1(u\alpha) = b(\alpha, u\alpha) = (b[u])\alpha$.

$$\frac{\Gamma \vdash}{1 : \Gamma \to \Gamma} \qquad \frac{\sigma : \Delta \to \Gamma \quad \delta : \Theta \to \Delta}{\sigma\delta : \Theta \to \Gamma} \qquad \frac{\Gamma \vdash A \quad \sigma : \Delta \to \Gamma}{\Delta \vdash A\sigma} \qquad \frac{\Gamma \vdash t : A \quad \sigma : \Delta \to \Gamma}{\Delta \vdash t\sigma : A\sigma}$$

$$\frac{}{() \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A}{\Gamma.A \vdash} \qquad \frac{\Gamma \vdash A}{\mathsf{p} : \Gamma.A \to \Gamma} \qquad \frac{\Gamma \vdash A}{\Gamma.A \vdash \mathsf{q} : A\mathsf{p}}$$

$$\frac{\sigma : \Delta \to \Gamma \quad \Gamma \vdash A \quad \Delta \vdash u : A\sigma}{(\sigma, u) : \Delta \to \Gamma.A}$$

$$\frac{\Gamma.A \vdash B}{\Gamma \vdash \Pi\ A\ B} \qquad \frac{\Gamma.A \vdash B \quad \Gamma.A \vdash b : B}{\Gamma \vdash \lambda b : \Pi\ A\ B} \qquad \frac{\Gamma \vdash w : \Pi\ A\ B \quad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}(w, u) : B[u]}$$

$$\frac{\Gamma.A \vdash B}{\Gamma \vdash \Sigma\ A\ B} \qquad \frac{\Gamma.A \vdash B \quad \Gamma \vdash u : A \quad \Gamma \vdash v : B[u]}{\Gamma \vdash (u, v) : \Sigma\ A\ B} \qquad \frac{\Gamma \vdash w : \Sigma\ A\ B}{\Gamma \vdash w.1 : A} \qquad \frac{\Gamma \vdash w : \Sigma\ A\ B}{\Gamma \vdash w.2 : B[w.1]}$$

$$1\sigma = \sigma 1 = \sigma \qquad (\sigma\delta)\nu = \sigma(\delta\nu) \qquad [u] = (1, u)$$

$$A1 = A \quad (A\sigma)\delta = A(\sigma\delta) \qquad u1 = u \quad (u\sigma)\delta = u(\sigma\delta)$$

$$(\sigma, u)\delta = (\sigma\delta, u\delta) \quad \mathsf{p}(\sigma, u) = \sigma \quad \mathsf{q}(\sigma, u) = u \quad (\mathsf{p}, \mathsf{q}) = 1$$

$$(\Pi\ A\ B)\sigma = \Pi\ (A\sigma)\ (B(\sigma\mathsf{p}, \mathsf{q})) \qquad (\lambda b)\sigma = \lambda(b(\sigma\mathsf{p}, \mathsf{q}))$$

$$\mathsf{app}(w, u)\delta = \mathsf{app}(w\delta, u\delta) \qquad \mathsf{app}(\lambda b, u) = b[u] \qquad w = \lambda(\mathsf{app}(w\mathsf{p}, \mathsf{q}))$$

$$(\Sigma\ A\ B)\sigma = \Sigma\ (A\sigma)\ (B(\sigma\mathsf{p}, \mathsf{q})) \quad (w.1)\sigma = (w\sigma).1 \quad (w.2)\sigma = (w\sigma).2$$

$$(u, v)\sigma = (u\sigma, v\sigma) \qquad (u, v).1 = u \qquad (u, v).2 = v \qquad (w.1, w.2) = w$$

Figure 1 Rules of MLTT.

The definition of dependent sums is easier: $\Gamma \vdash \Sigma\, A\, B$ for $\Gamma \vdash A$ and $\Gamma.A \vdash B$ is defined by sums in each stage, i.e. for an I-cube α in Γ, $(\Sigma\, A\, B)\alpha$ consists of pairs (u, v) with u in $A\alpha$ and v in $B(\alpha, u)$. Restrictions are defined component-wise: $(u, v)f = (uf, vf)$ where $f \colon I \to J$. If $\Gamma \vdash w : \Sigma\, A\, B$ and $w\alpha = (u, v)$, then $(w.1)\alpha = u$ and $(w.2)\alpha = v$.

We can then verify all the equations of Figure 1.

5 The uniform Kan condition

Using these notations we can formulate the Kan condition (cf. [15]) and our strengthening as follows. Let X be a cubical set. First we define the notion of an *open box* in X, the equivalent of a *horn* in a simplicial set. Let I be a finite set of names and let $J, x \subseteq I$. The variable x must not be in J and will be the direction in which the box is open. For every $y \in J$, the open box will have two faces, one with $y = 0$ and one with $y = 1$. Let $O^+(J, x)$ consist of pairs $(x, 0)$ and (y, b) for $y \in J,\ b = 0, 1$. In the same way we define $O^-(J, x)$, but with $(x, 1)$ instead of $(x, 0)$. The idea for both is that one face in the direction x is missing. We use $O(J, x)$ to denote either $O^+(J, x)$ or $O^-(J, x)$. An *open box*, denoted by $\vec{u}$, is a family of elements (faces) u_{yb} in $X(I - y)$ for each $(y, b) \in O(J, x)$ such that

$$u_{yb}(z = c) = u_{zc}(y = b)$$

for all $(y, b), (z, c) \in O(J, x)$ with $y \neq z$. The latter condition may be phrased as: *the faces of an open box are adjacent-compatible.* If $f : I \to K$ is defined on J, x, we write $\vec{u}f$ for the open box indexed by $O(f(J), f(x))$ with components $(\vec{u}f)_{(fy)b} = u_{yb}(f - y)$ in $X(K - f(y))$.

For X to be a *(constructive) Kan cubical set*, we require to be given operations $X{\uparrow}$ and $X{\downarrow}$ for every $J, x \subseteq I$ such that $X{\uparrow}\vec{u}$ and $X{\downarrow}\vec{u}$ are both in $X(I)$. Here $\vec{u}$ is an open box with u_{x0} and u_{x1} in $X(I - x)$ in the respective cases $X{\uparrow}\vec{u}$ and $X{\downarrow}\vec{u}$. (From now on we will always tacitly assume that the open box $\vec{u}$ is of the right type with respect to $X{\uparrow}, X{\downarrow}$.) The operations $X{\uparrow}, X{\downarrow}$ are to be thought of as a filling their respective open boxes. Therefore we require for all $(y, b) \in O(J, x)$:

$$(X{\uparrow}\vec{u})(y = b) = u_{yb} \qquad (X{\downarrow}\vec{u})(y = b) = u_{yb}$$

The new uniformity condition is: if $f : I \to K$ is defined on J, x, we require:

$$(X{\uparrow}\vec{u})f = X{\uparrow}(\vec{u}f) \qquad (X{\downarrow}\vec{u})f = X{\downarrow}(\vec{u}f)$$

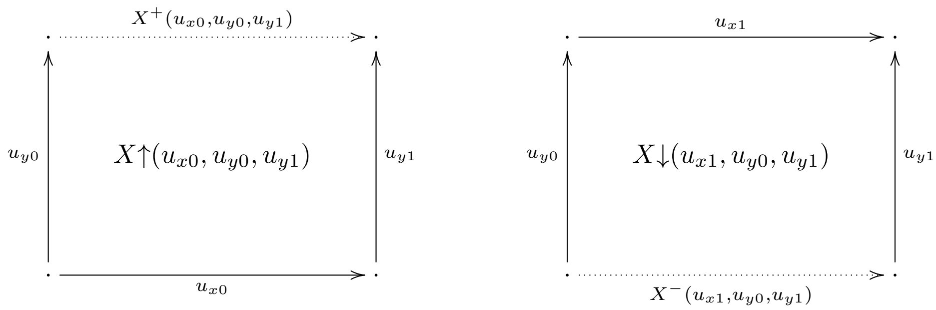
We refer to the combined condition as the *uniform Kan condition for cubical sets*, or the *Kan condition* for short.

If we only consider the case where $I = J, x$, that is, no other variables in I, and without the uniformity conditions, we get back the usual notion of Kan cubical sets [15, Sec. 4] (adapted to our notion of cubical sets). Similar uniformity conditions have been considered for semisimplicial sets in [2]. For a suggestive description of how to define combinatorially $\pi_n(X, u)$ for each point u of X if X satisfies the Kan property, see [27].

If X is a Kan cubical set with operations $X{\uparrow}, X{\downarrow}$, we define new operations (see the figure below)

$$X^+\vec{u} = (X{\uparrow}\vec{u})(x = 1) \qquad X^-\vec{u} = (X{\downarrow}\vec{u})(x = 0)$$

representing *transport* in the open box in the direction in which it is open.



Let Γ be a cubical set (which does not need to satisfy the Kan condition) and $\Gamma \vdash A$ a type over Γ. A *Kan structure* on $\Gamma \vdash A$ is given by operations $A\alpha{\uparrow}$ and $A\alpha{\downarrow}$ for each $\alpha \in \Gamma(I)$ and $J, x \subseteq I$, such that $A\alpha{\uparrow}\vec{u}$ and $A\alpha{\downarrow}\vec{u}$ are both in $A\alpha$ for every open box $\vec{u}$. Here *open box* means that $u_{yb} \in A\alpha(y = b)$ for all $(y, b) \in O(J, x)$, and that these faces are adjacent-compatible. $A\alpha{\uparrow}, A\alpha{\downarrow}$ must satisfy the same Kan conditions as $X{\uparrow}, X{\downarrow}$ above. The usual Kan conditions are obtained by simply substituting $A\alpha$ for X. Since $f : I \to K$ interacts with α, we reformulate the uniformity conditions:

$$(A\alpha{\uparrow}\vec{u})f = A\alpha f{\uparrow}(\vec{u}f) \qquad (A\alpha{\downarrow}\vec{u})f = A\alpha f{\downarrow}(\vec{u}f)$$

If $\Gamma \vdash A$ has a Kan structure with operations $A\alpha{\uparrow}, A\alpha{\downarrow}$, we define as before

$$A\alpha^+\vec{u} = (A\alpha{\uparrow}\vec{u})(x = 1) \qquad A\alpha^-\vec{u} = (A\alpha{\downarrow}\vec{u})(x = 0)$$

Notice that if $\Gamma \vdash A$ has a Kan structure, then the map $\mathsf{p}\colon \Gamma.A \to \Gamma$ is a Kan fibration as in [15, 27].

For $\Gamma \vdash A$ with Kan structure and a line α in $\Gamma(x)$ connecting points ρ_0 to ρ_1 one can define a map of cubical sets $A\rho_0 \to A\rho_1$ as follows. First, consider $A\rho_i$ as a cubical set with set of points $A\rho_i$, set of lines $A\rho_i\iota_x$, and so on. In general, we define $A\rho_i\iota_I$ by taking ι_I to be the unique morphism $\emptyset \to I$; restrictions are induced by $\Gamma \vdash A$. Then, the map $A\rho_0 \to A\rho_1$ is defined by $a \mapsto A\alpha^+a$. The equivalence $a \mapsto A\alpha^+a$ works uniformly and does not distinguish cases in which a is degenerate or not. One can show that this map is an equivalence (see Section 8.2 and 8.4). This is in contrast to Kan simplicial sets where classical logic is essential to define such an equivalence [3].

6 Examples of cubical sets

In this section we elaborate the following examples of cubical sets: discrete cubical sets; the unit interval $\mathbb{I}$; polynomial rings; the cubical nerve N of the group Z_2 with two elements; the exponential $N^{\mathbb{I}}$. A noticeable difference between simplicial sets and our cubical sets is that, while N is Kan, $N^{\mathbb{I}}$ is not. This is important motivation for the main result of the next section, implying that B^A is a Kan cubical set if both A and B are.

Every set A gives rise to the *discrete cubical set* $\mathsf{K}\,A$ via the constant presheaf, i.e. $(\mathsf{K}\,A)(I) = A$ for each I and all restrictions are the identity map $A \to A$. Note that in an open box $\vec{u}$ all the components have to be equal, say u, and this u is also the (unique) filler $u = \mathsf{K}\,A{\uparrow}\vec{u}$ making the discrete cubical set trivially into a Kan cubical set.

6.1 Unit interval

Recall the canonical extension of a map $f : J \to K$ in $\mathcal{C}$ to a set map $J \cup \{0,1\} \to K \cup \{0,1\}$ that is the identity on $\{0,1\}$. Together with mapping objects J of $\mathcal{C}$ to $J \cup \{0,1\}$, canonical extension actually forms a functor $\mathcal{C} \to \mathsf{Set}$. This covariant functor is called the *unit interval*, denoted by $\mathbb{I}$. We explore: $\mathbb{I}() = \{0,1\}$ ($\mathbb{I}$ has two points); $\mathbb{I}(x) = \{0,1,x\}$ ($\mathbb{I}$ has three lines, only x is non-degenerate); $\mathbb{I}(x,y) = \{0,1,x,y\}$ ($\mathbb{I}$ has four degenerate squares, see the display below); and so on. The square x varies in direction x, but is constant in direction y, and hence degenerate. Similarly for objects of higher dimension in $\mathbb{I}$. This completes the description of the unit interval as a cubical set. Note that $\mathbb{I} \cong \mathbf{y}\{x\}$ for a name x (where $\mathbf{y}$ denotes the Yoneda embedding) is another way to describe the interval.

$$\mathbb{I}(x,y) : \begin{matrix} 0\ 0 \\ 0\ 0 \end{matrix} \quad \begin{matrix} 1\ 1 \\ 1\ 1 \end{matrix} \quad \begin{matrix} 0\ 1 \\ 0\ 1 \end{matrix} \quad \begin{matrix} 1\ 1 \\ 0\ 0 \end{matrix}$$

6.2 Polynomial rings

A particularly natural example of a cubical set, suggested by Aczel, is based on polynomials over a commutative ring R. We let $R[I]$ as usual denote the ring of polynomials with coefficients in R and variables in (at most) I. For $x \notin I$ and $p \in R[I]$, we define $p\iota_x = p \in R[I,x]$. For $x \in I$ and $p \in R[I]$, we define $p(x=0) \in R[I-x]$ as p with $0 \in R$ substituted for x. Likewise, $p(x=1) \in R[I-x]$ is p with $1 \in R$ substituted for x. It is easily verified that this defines a cubical set, which we denote by $R[\_]$. In the following paragraph we show that $R[\_]$ has Kan structure.

For simplicity, we first take $I = x, J$ and $J = y, z$. After that, the general case can be done by an easy induction on $|J|$. Let $\vec{u}$ be an open box indexed by $O^+(J,x)$. The construction of filling an open box can be described as *iterated orthogonal linear interpolation*, in which we stepwise approximate the filler p, one direction per step, ending with the direction in which the box is open. Define $p_z = (1-z)u_{z0} + zu_{z1}$. Then $p_z(z=0) = u_{z0}$, $p_z(z=1) = u_{z1}$, so p_z has the right faces in direction z. Now define:

$$p_{yz} = p_z + (1-y)(u_{y0} - p_z(y=0)) + y(u_{y1} - p_z(y=1))$$

This step is typical for the induction case. Per construction, p_{yz} has the right faces in the direction y. We verify that p_{yz} still has the right faces in the direction z. For $b = 0,1$ we have

$$\begin{aligned} p_{yz}(z=b) &= p_z(z=b) + (1-y)(u_{y0}(z=b) - p_z(z=b)(y=0)) \\ &\quad + y(u_{y1}(z=b) - p_z(z=b)(y=1)) \\ &= u_{zb} + (1-y)(u_{y0}(z=b) - u_{zb}(y=0)) + y(u_{y1}(z=b) - u_{zb}(y=1)) \\ &= u_{zb}. \end{aligned}$$

In the last step above we have used that an open box has adjacent-compatible faces, such that $u_{yc}(z=b) = u_{zb}(y=c)$. It remains to define the filler $p = p_{xyz}$ by

$$p_{xyz} = p_{yz} + (1-x)(u_{x0} - p_{yz}(x=0))$$

and to verify the p has all the same faces as $\vec{u}$. The latter is similar to the verification of p_{yz} and is left to the reader. We note that the construction of the filler p is completely uniform, and hence $R[\_]$ has Kan structure.

We can also fill closed boxes by adding a term $x(u_{x1} - p_{yz}(x=1))$ to p_{xyz} above. Another consequence of linear interpolation is that the cubical set $R[\_]$ is contractible.

6.3 Cubical nerve

Recall that a morphism $f : J \to K$ in $\mathcal{C}$ is a function $f : J \to K \cup \{0,1\}$ such that for every $y \in K$ there exists at most one $x \in J$ with $f(x) = y$. Hence every morphism $f : J \to K$ defines a function $\{0,1\}^K \to \{0,1\}^J$ through precomposition with f. We can view $\{0,1\}^J$ as a product of posets $0 \leq 1$, and hence as a category with unique morphisms. Then every morphism $f : J \to K$ defines a functor $\{0,1\}^K \to \{0,1\}^J$, as the precomposition preserves the order. We denote this functor also by f.

Given a small category $\mathcal{D}$, we define its *cubical nerve* $N\mathcal{D}$ as follows. The sets $N\mathcal{D}(J)$ are functors $\{0,1\}^J \to \mathcal{D}$. For every morphism $f : J \to K$, its function $N\mathcal{D}(J) \to N\mathcal{D}(K)$ is defined by precomposition with the functor f. Note that the unit interval is not the cubical nerve of the poset $0 \leq 1$: they have similar sets of points and lines, but $N(0 \leq 1)$ has two more squares, both non-degenerate in two directions:

$$N(0 \leq 1)(x,y) : \begin{matrix} 0\ 0 \\ 0\ 0 \end{matrix} \quad \begin{matrix} 0\ 1 \\ 0\ 1 \end{matrix} \quad \begin{matrix} 1\ 1 \\ 0\ 0 \end{matrix} \quad \begin{matrix} 1\ 1 \\ 1\ 1 \end{matrix} \quad \begin{matrix} 0\ 1 \\ 0\ 0 \end{matrix} \quad \begin{matrix} 1\ 1 \\ 0\ 1 \end{matrix}$$

An element of $N\mathcal{D}(J)$ can be viewed as a (hyper)cube with the edges labelled by morphisms of $\mathcal{D}$ and vertices labelled by objects of $\mathcal{D}$, such that all paths commute (or equivalently, all triangles commute). This completes the description of the cubical nerve of a category.

Consider the group of two elements as a category (groupoid) with one object $\star$ and two morphisms $0,1 : \star \to \star$ where 0 is the identity of $\star$. Let N be the nerve of this groupoid: N has one point and two lines, again denoted by $\star$ and $0,1$. Note that $\star\iota_x = 0$ and $1 \circ 1 = 1 + 1 = 0$. The squares of N are listed as follows, where we only show the lines:

$$N(x,y)\colon\ {}_{0}\overset{0}{\underset{0}{\ }}{}_{0} \quad {}_{1}\overset{0}{\underset{0}{\ }}{}_{1} \quad {}_{0}\overset{1}{\underset{1}{\ }}{}_{0} \quad {}_{1}\overset{0}{\underset{1}{\ }}{}_{0} \quad {}_{1}\overset{1}{\underset{0}{\ }}{}_{0} \quad {}_{0}\overset{1}{\underset{0}{\ }}{}_{1} \quad {}_{0}\overset{0}{\underset{1}{\ }}{}_{1} \quad {}_{1}\overset{1}{\underset{1}{\ }}{}_{1}$$

Being the nerve of a groupoid, N is Kan (see the next section).

We now show that $N^{\mathbb{I}}$ is not Kan. By the Yoneda Lemma we have $N^{\mathbb{I}}(J) \cong ((\mathbf{y}J \times \mathbb{I}) \to N)$, the latter denoting a set of natural transformations. Defining $p \in N^{\mathbb{I}}(J)$ means defining maps (index K omitted) $p : \mathbf{y}J(K) \to (\mathbb{I}(K) \to N(K))$ for all K, such that $(p_f u)g = p_{fg}(ug)$ for every $f : J \to K$, $g : K \to L$, and $u \in \mathbb{I}(K)$.

We explore the points of $N^{\mathbb{I}}$ and define $p \in N^{\mathbb{I}}()$ by, first $p_{1_{()}} : \mathbb{I}() \to N() : 0,1 \mapsto \star$. Then, $p_{\iota_x} : \mathbb{I}(x) \to N(x) : 0,1 \mapsto \star\iota_x = 0$ is forced by naturality, but for $p_{\iota_x} x$ there is a choice. If we choose 0, we must make the same choice for all names x in the name space. The choice 1 for all names x in the name space would give the only other point. In higher dimensions all arguments are degenerate, determining the function values, and naturality is compatible with each of the two choices above. We now fix p with $p_{\iota_x} x = 0$.

Next we explore lines from p to p in $N^{\mathbb{I}}$, say in direction i, and define $\ell : p \to p$ in $N^{\mathbb{I}}(i)$ by $\ell_{(i=b)g} = p_g$ for all $b = 0,1$ and $g : \emptyset \to K$. For $\ell_{(i=x)} : \mathbb{I}(x) \to N(x)$ there is a choice. For the moment we put $\ell_{(i=x)} c = \ell_c$ for all $c \in \mathbb{I}(x)$. Note that we must make the same choices ℓ_0, ℓ_1, ℓ_x for all names x in the name space. On the next level, there is no choice left. First, $\ell_{(i=b)g} = p_g$ for $b = 0,1$ and $g = \iota_x \iota_y$. Moreover, $\ell_{(i=x)\iota_y}, \ell_{(i=y)\iota_x} : \mathbb{I}(x,y) \to N(x,y)$ are completely determined by the choices of ℓ_0, ℓ_1, ℓ_x. Even more so, naturality limits the choice on the lower level. This can be seen by applying $\ell_{(i=x)\iota_y}$ and $\ell_{(i=y)\iota_x}$ to both x and y in $\mathbb{I}(x,y)$. This results in the four squares (NB: $\ell_x = \ell_y$):

$$\begin{array}{ccc} & \ell_x & \\ 0 & & 0 \\ & \ell_x & \end{array} \quad \begin{array}{ccc} & \ell_1 & \\ 0 & & 0 \\ & \ell_0 & \end{array} \quad \begin{array}{ccc} & 0 & \\ \ell_0 & & \ell_1 \\ & 0 & \end{array} \quad \begin{array}{ccc} & 0 & \\ \ell_y & & \ell_y \\ & 0 & \end{array}$$

Since the squares have to commute we get $\ell_0 = \ell_1$. In higher dimensions all values are determined by naturality, and naturality is compatible with each of the four possible choices (recall that objects in $\mathbb{I}$ can be non-degenerate in at most one direction). This yields in total four lines from p to p in $N^{\mathbb{I}}$.

In order to show that $N^{\mathbb{I}}$ is not Kan, consider lines $p, \ell : p \to p$, where p is degenerate ($p_0 = p_x = p_1 = 0$) and ℓ is defined by $\ell_0 = \ell_1 = 0,\ \ell_x = 1$. Consider an open box as in the picture below, left:

$$\begin{tikzcd} p \arrow[r, dotted] & p \\ p \arrow[u, "p"] \arrow[r, "p"'] & p \arrow[u, "\ell"'] \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r, "0"] & 0 \\ 0 \arrow[u, "0"] \arrow[r, "0"'] & 0 \arrow[u, "0"'] \end{tikzcd} \quad \begin{tikzcd} \star \arrow[r, "\ell'_0"] & \star \\ \star \arrow[u, "0"] \arrow[r, "0"'] & \star \arrow[u, "0"'] \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r, "1"] & 1 \\ 0 \arrow[u, "y"] \arrow[r, "0"'] & 0 \arrow[u, "y"'] \end{tikzcd} \quad \begin{tikzcd} \star \arrow[r, "\ell'_1"] & \star \\ \star \arrow[u, "0"] \arrow[r, "0"'] & \star \arrow[u, "1"'] \end{tikzcd}$$

Assume we could fill the box. Call the closing (dotted) line above ℓ'. Applying the first square to the second results in the third square, yielding $\ell'_0 = 0$. Applying the first square to the fourth results in the last square, yielding $\ell'_1 = 1$. This contradicts $\ell'_0 = \ell'_1$ for any line $p \to p$. Hence the above box has no filler.

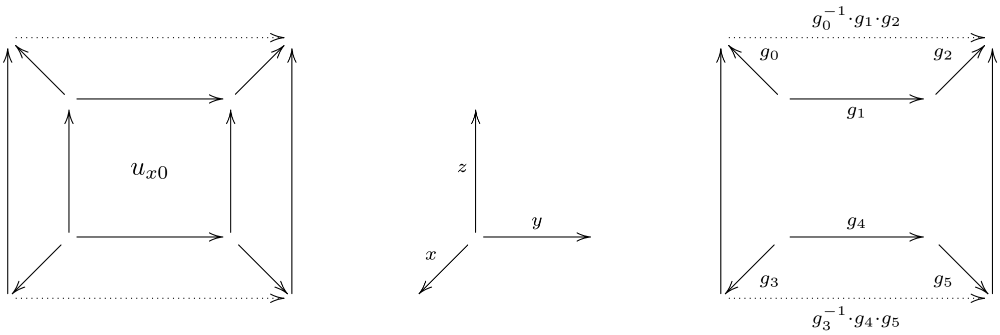
6.4 The nerve of a groupoid is Kan

Let G be a groupoid, and N its cubical nerve. We sketch a proof that N is Kan. Take $I = x, J, z$ in $\mathcal{C}$, with $J = y_1, \ldots, y_k$ ($k \geq 0$). Taking one variable z instead of $z_1, \ldots, z_n$ simplifies the presentation, but is otherwise inessential.

Let $\vec{u}$ be an open box indexed by $O(J,x)$, that is, adjacent-compatible faces $u_{x0} \in N(I-x)$ and u_{yb} in $N(I-y)$. We have to define $u \in N(I)$ with faces as given by the open box. For this we define closing faces u_{x1},u_{z0},u_{z1}, such that they are adjacent-compatible with the open box, and show that all squares commute. This will define u in a unique way. Thereafter we shall verify the uniformity condition.

If $J = \emptyset$ ($k = 0$), the open 'box' is a degenerate line u_{x0} in direction z. We close by taking $u_{x1} = u_{z0} = u_{z1} = u_{x0}$, and u is the doubly degenerate square. If $J \neq \emptyset$ ($k > 0$), we observe that all the points of u are already given by the open box, so that we can limit attention to the edges. Moreover, if J consists of more than one variable, all edges are also already present in the open box, which makes the definition of the closing faces particularly simple. This can be seen as follows. For $b = 0, 1$, the faces $u_{y_1 b}$ contain all edges in which $y_1 = b$, and the faces $u_{y_2 b}$ contain all edges in which $y_2 = b$. In particular, the two faces $u_{y_2 b}$ contain all edges in direction y_1. Hence, the four faces $u_{y_1 b}$, $u_{y_2 b}$ together contain all edges. The groupoid structure guarantees that all squares of the closing face commute.

The most interesting case to elaborate is $I = x, y, z,\ \ J = y$, where we have to define the edges in u_{x1} in direction y. This situation is depicted below, left, with the new edges as defined right. The new edges make essential use of the inverses in the groupoid and are uniquely defined.



The new squares u_{zb} commute as per construction. Moreover, the new square u_{x1} commutes since it can be projected down to the commuting square u_{x0} along edges that are invertible. A similar argument can be used if J contains more variables. This completes the construction of $u \in N(I)$.

Uniformity will be shown to be a consequence of the uniqueness of u constructed above, and the following easy lemma. This lemma can be useful in other places as well.

▶ **Lemma 3.** *For all morphisms $f : I \to K$ in $\mathcal{C}$ defined on x we have (i) $(x = b)(f - x) = f(f(x) = b)$ and (ii) $\iota_x f = (f - x)\iota_{fx}$.*

Now let $u = N{\uparrow}(u_{x0}, u_{y0}, u_{y1})$ and $u' = N{\uparrow}(u_{x0}(f - x), u_{y0}(f - y), u_{y1}(f - y))$. We have to show $u' = uf$. By the lemma we have $u_{x0}(f - x) = uf(f(x) = 0)$ and $u_{yb}(f - y) = uf(f(y) = b)$. This means that u' and uf agree on the open box defining u', so they are equal by uniqueness. Again, a similar argument can be used if J contains more variables. This completes the proof sketch that the cubical nerve of a groupoid is Kan.

7 The Kan cubical set model

In this section we will give a refinement of the model given in Section 4. The *Kan cubical set model* is given as follows: contexts and context morphisms are interpreted as in Section 4, i.e. by cubical sets and morphisms between cubical sets; a type is given by a type $\Gamma \vdash A$ in the sense of Section 4 *together* with a Kan structure; terms are given as in Section 4. The Kan structure on types is needed in order to justify the elimination rules for the identity types (cf. Section 8.2).

It is crucial to note that the Kan structure is part of a type in the Kan cubical set model. Two types $\Gamma \vdash A$ and $\Gamma \vdash B$ which have a Kan structure can be equal as cubical sets, but *not* with their Kan structure. Thus we have to check whether the equations between types in Figure 1 are preserved *for their Kan structure.*

The definition of the model is such that it follows the model described in Section 4, but additionally we have to define how the Kan structure is given on the types. This is done in the proofs of the following theorems.

▶ **Theorem 4.** *If $\Gamma \vdash A$ has a Kan structure and $\sigma\colon \Delta \to \Gamma$, then also $\Delta \vdash A\sigma$ has a Kan structure. Moreover the definition is such that $A1 = A$ and $(A\sigma)\tau = A(\sigma\tau)$ as types with Kan structures.*

Proof. For an I-cube α of Δ recall that $(A\sigma)\alpha = A(\sigma\alpha)$ as cubical sets; we define the filling operations in $(A\sigma)\alpha$ to be those in $A(\sigma\alpha)$, i.e. we set $(A\sigma)\alpha{\uparrow}\vec{u} = A(\sigma\alpha){\uparrow}\vec{u}$. With this definition it is clear that $A1$ and A have the same filling operations, and similarly for the other equation. ◀

7.1 Dependent product

▶ **Theorem 5.** *If both $\Gamma \vdash A$ and $\Gamma.A \vdash B$ have Kan structures, then so does $\Gamma \vdash \Pi\, A\, B$. Moreover the definition of the Kan structure is such that $(\Pi\, A\, B)\sigma = \Pi(A\sigma)(B(\sigma\mathsf{p},\mathsf{q}))$.*

Proof. We present the argument in the case $J = \emptyset$, the general case is not essentially more difficult. Also, as the cases $\uparrow, \downarrow$ are perfectly symmetric, we restrict attention to $\uparrow$. We denote the direction of filling with a subscript to $\uparrow, \downarrow, ^-, ^+$. Let $C = \Pi\, A\, B$.

First we will define $C\alpha^+_x w \in C\alpha(x=1)$ for α an I-cube of Γ, $x \in I$, and w in $C\alpha(x=0)$. This amounts to define a family of dependent functions $(C\alpha^+_x w)_f$ in $\prod_{u \in A\alpha(x=1)f} B(\alpha(x=1)f, u)$ for all $f\colon I - x \to K$, such that

$$\big((C\alpha^+_x w)_f(u)\big)g = (C\alpha^+_x w)_{fg}(ug). \tag{1}$$

We will first define $(C\alpha^+_x w)_f$ for $f = 1\colon I - x \to I - x$. For this let $u \in A\alpha(x=1)$. We use the Kan fillings to map u down to $A\alpha^-_x u$, apply w (at $1\colon I - x \to I - x$) and map the result up:

$$(C\alpha^+_x w)_1(u) = B(\alpha, A\alpha{\downarrow}_x u)^+_x(w_1(A\alpha^-_x u)) \tag{2}$$

which is in $B(\alpha(x=1), u)$ as $(A\alpha{\downarrow}_x u)(x=1) = u$. So we have defined $(C\alpha^+ w)_1$ for arbitrary α and w.

For general $f\colon I - x \to K$ we let z be fresh w.r.t. K and set:

$$(C\alpha^+_x w)_f = (C\alpha(f, x=z)^+_z wf)_1 \tag{3}$$

By the uniformity conditions, this definition does not depend on the choice of z, and we also get by uniformity and (2)

$$\big((C\alpha^+_x w)_1(u)\big)f = (C\alpha(f, x=z)^+_z wf)_1(uf). \tag{4}$$

Note that (3) suffices to get the uniformity conditions for $C\alpha^+_x w$; (3) together with (4), yields (1) and thus an element in $C\alpha(x=1)$, concluding the definition of $C\alpha^+_x w$.

Next we define $C\alpha{\uparrow}_x w \in C\alpha$; we do so again by first defining $(C\alpha{\uparrow}_x w)_f$ for $f = 1\colon I \to I$. Let $\gamma \in A\alpha$, $u_0 = \gamma(x=0)$ and $u = \gamma(x=1)$; the definition of $(C\alpha{\uparrow}_x w)_1(\gamma) \in B(\alpha, \gamma)$ has to satisfy:

$$
\begin{tikzcd}
(C\alpha^+_x w)_1 & : & u & \longmapsto & (C\alpha^+_x w)_1(u) \\
(C\alpha{\uparrow_x} w)_1 \arrow[u, dotted] & : & {} & \longmapsto & (C\alpha{\uparrow_x} w)_1(\gamma) \arrow[u, dotted] \\
w_1 \arrow[u, dotted] & : & u_0 \arrow[uu, "\gamma"'] & \longmapsto & w_1(u_0) \arrow[u, dotted]
\end{tikzcd}
$$

Let y be a fresh name; using the uniform Kan filling for $\Gamma \vdash A$ in $A\alpha$ with $J = \{y\}$ (denoted by $A\alpha{\downarrow_{x,y}}$) we construct

$$\theta = A\alpha{\downarrow_{x,y}}(u, \gamma, A\alpha{\downarrow_x} u),$$

resulting in a square:

$$
\begin{tikzcd}
u \arrow[r, "u"] & u \\
u_0 \arrow[u, "\gamma"] \arrow[r, dotted, "\theta(x=0)"'] \arrow[ur, phantom, "\theta"] & A\alpha^-_x u \arrow[u, "A\alpha{\downarrow_x} u"']
\end{tikzcd}
$$

With $\lambda = B(\alpha, A\alpha{\downarrow_x} u){\uparrow_x}(w_1(A\alpha^-_x u))$ we get an open box in $B(\alpha, \theta)$

$$
\begin{tikzcd}
(C\alpha^+_x w)_1(u) \arrow[r, "(C\alpha^+_x w)_1(u)"] & (C\alpha^+_x w)_1(u) \\
w_1 u_0 \arrow[r, "w_{\iota_y}(\theta(x=0))"'] & w_1(A\alpha^-_x u) \arrow[u, "\lambda"']
\end{tikzcd}
$$

where the line on the right hand side is by the defining equation (2). Using the Kan structure of $\Gamma.A \vdash B$ for $J = \{x\}$ we define

$$(C\alpha{\uparrow_x} w)_1(\gamma) = B(\alpha, \theta)^-_{y,x}\big(\lambda, w_{\iota_y}(\theta(x = 0)), (C\alpha^+_x w)_1(u)\big)$$

with λ as above. Using the uniformity conditions for $\Gamma \vdash A$ and $\Gamma.A \vdash B$, this definition is such that

$$\big((C\alpha{\uparrow_x} w)_1(\gamma)\big)f = (C\alpha f{\uparrow_{fx}} w(f - x))_1(\gamma f)$$

for $f \colon I \to K$ defined on x.

Now, if $f : I \to K$ is defined on x, we define $(C\alpha{\uparrow} w)_f = (C\alpha f{\uparrow_{fx}} w(f - x))_1$. If f is not defined on x, we can write $f = (x = b)f'$ for some $f' : I - x \to K$. Then we can simply define $(C\alpha{\uparrow_x} w)_f = w_{f'}$ for $b = 0$, and $(C\alpha{\uparrow_x} w)_f = (C\alpha^+_x w)_{f'}$ for $b = 1$. This defines the element $C\alpha{\uparrow} w$ in $C\alpha$ which satisfies the uniformity conditions.

To verify that the Kan structure of $\Pi(A\sigma)(B(\sigma\mathsf{p}, \mathsf{q}))$ (as defined above) is equal to the Kan structure for $(\Pi\, A\, B)\sigma$ (as defined in the proof of the preceding theorem), assume that above $\alpha = \sigma\beta$ for $\sigma \colon \Delta \to \Gamma$; then $C\alpha = ((\Pi\, A\, B)\sigma)\beta$ and in equation (2) we have

$$B(\sigma\beta, A(\sigma\beta){\downarrow_x} u)^+_x(w_1(A(\sigma\beta)^-_x u)) = (B(\sigma\mathsf{p}, \mathsf{q}))(\beta, (A\sigma)\beta{\downarrow_x} u)^+_x(w_1((A\sigma)\beta^-_x u))$$

and the right hand side is the definition of $\big(\Pi(A\sigma)(B(\sigma\mathsf{p}, \mathsf{q}))^+_x w\big)_1(u)$. Similarly for the other parts of the definition. ◀

Notice that we make essential use of the uniformity conditions in the above proof in order to verify that the fillers we define are indeed elements in the dependent product. Moreover, in the general case the fillings used from $\Gamma \vdash A$ are only with J such that $|J| \leq 1$.

7.2 Sum type

▶ **Theorem 6.** *If* $\Gamma \vdash A$ *and* $\Gamma.A \vdash B$ *have Kan structures, then so does* $\Gamma \vdash \Sigma\, A\, B$*. Moreover the definition of the Kan structure is such that* $(\Sigma\, A\, B)\sigma = \Sigma(A\sigma)(B(\sigma\mathsf{p},\mathsf{q}))$*.*

Proof. Given an open box $\vec{p}$ in $(\Sigma\, A\, B)\alpha$ with $p_{yb} = (u_{yb}, v_{yb})$ for any $(y, b) \in O^+(J, x)$ we first fill $u = A\alpha{\uparrow}\vec{u}$ in $A\alpha$, and then set

$$(\Sigma\, A\, B)\alpha{\uparrow}\vec{p} = (u, B(\alpha, u){\uparrow}\vec{v}).$$

This clearly satisfies the uniformity condition as they are satisfied for $\Gamma \vdash A$ and $\Gamma.A \vdash B$.

Moreover, if $\alpha = \sigma\beta$ for $\sigma \colon \Delta \to \Gamma$, we get $u = (A\sigma)\beta{\uparrow}\vec{u}$ and

$$B(\sigma\beta, u){\uparrow}\vec{v} = (B(\sigma\mathsf{p},\mathsf{q}))(\beta, u){\uparrow}\vec{v},$$

yielding $(\Sigma\, A\, B)\sigma = \Sigma(A\sigma)(B(\sigma\mathsf{p},\mathsf{q}))$. ◀

8 Extensions

8.1 Inductive types

We can interpret inductive types by adding the corresponding constructors in each dimension. In the case of inductive definitions without parameters this results in a discrete Kan cubical sets (see Section 6). E.g. the booleans $\Gamma \vdash N_2$ are defined by $N_2\alpha = \{\mathsf{true}, \mathsf{false}\}$ for each $\alpha \in \Gamma(I)$, and restrictions being the identity map. As in Section 6 one defines a Kan structure. We interpret the constants $\Gamma \vdash \mathsf{true} : N_2$ by $\mathsf{true}\alpha = \mathsf{true}$, and similar for $\Gamma \vdash \mathsf{false} : N_2$. To interpret the elimination principle

$$\frac{\Gamma.N_2 \vdash C \quad \Gamma \vdash d_0 : C[\mathsf{true}] \quad \Gamma \vdash d_1 : C[\mathsf{false}] \quad \Gamma \vdash b : N_2}{\Gamma \vdash \mathsf{if}\ b\ \mathsf{then}\ d_0\ \mathsf{else}\ d_1 : C[b]}$$

we define $(\mathsf{if}\ b\ \mathsf{then}\ d_0\ \mathsf{else}\ d_1)\alpha = d_0\alpha$ for $b\alpha = \mathsf{true}$, and $(\mathsf{if}\ b\ \mathsf{then}\ d_0\ \mathsf{else}\ d_1)\alpha = d_1\alpha$ for $b\alpha = \mathsf{false}$.

8.2 Identity type

We describe the interpretation of $\Gamma \vdash \mathsf{Id}_A\ a\ b$ given $\Gamma \vdash A$ and $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$. Given an I-cube α in Γ we define $(\mathsf{Id}_A\ a\ b)\alpha$ to be the set of elements $\langle x\rangle\omega$ where ω is in $A\alpha\iota_x$ and x is a fresh variable not in I, such that $\omega(x = 0) = a\alpha$ and $\omega(x = 1) = b\alpha$. The latter situation is conveniently described by $\omega : a\alpha \to_x b\alpha$. We recall that ι_x denotes the canonical injection $I \to I, x$. The element $\langle x\rangle\omega$ is the equivalence class of I, x-cubes of $A\alpha\iota_x$, x not in I, where ω is identified with $\omega(x = x')$ if x' is not in I. This operation $\langle x\rangle\omega$ binds the name x. (One could define $\langle x\rangle\omega$ to be $\omega(x = x_I)$ where x_I is a name not in I obtained by a choice function.) If f is a substitution $I \to K$ we choose a variable y not in K, extend f to $(f, x = y) : I, x \to K, y$ and define $(\langle x\rangle\omega)f$ to be $\langle y\rangle\omega(f, x = y)$, preserving equivalence.

▶ **Theorem 7.** *If* $\Gamma \vdash A$ *has a Kan structure, then so does* $\Gamma \vdash \mathsf{Id}_A\ a\ b$ *whenever we have* $\Gamma \vdash a : A$ *and* $\Gamma \vdash b : A$*. Moreover the definition is such that* $(\mathsf{Id}_A\ a\ b)\sigma = \mathsf{Id}_{A\sigma}\ a\sigma\ b\sigma$ *as types with Kan structures.*

Proof. Let α be an I-cube of Γ and $J, x \subseteq I$. After a suitable renaming, we can conveniently denote an open box for $(\mathsf{Id}_A\ a\ b)\alpha$ by a vector $\langle y\rangle\vec{\omega}$ with components $\langle y\rangle\omega_{zc}$ in $(\mathsf{Id}_A\ a\ b)\alpha(z = c)$, for all $(z, c) \in O(J, x)$.

We define, with $a\alpha, b\alpha$ the faces in the direction y, omitting subscripts J,

$$(\mathsf{Id}_A\ a\ b)\alpha{\uparrow_x}\langle y\rangle\vec{\omega} = \langle y\rangle(A\alpha{\uparrow_{x,y}}(\vec{\omega}, a\alpha, b\alpha))$$

which shows that $\Gamma \vdash \mathsf{Id}_A\ a\ b$ satisfies the Kan condition for J, x if $\Gamma \vdash A$ satisfies the Kan condition for $(J, y), x$. The situation in case $J = \emptyset$ is depicted below. The uniformity condition follows from the uniformity of $\Gamma \vdash A$. ◀

$$\begin{tikzcd}
a\alpha \arrow[r, dotted, "A\alpha^+_{x,y}(\omega_0,a\alpha,b\alpha)"] & b\alpha & & \langle y\rangle\omega_1 \\
a\alpha \arrow[u, "a\alpha"] \arrow[r, "\omega_0"'] \arrow[ur, phantom, "A\alpha\uparrow_{x,y}(\omega_0,a\alpha,b\alpha)"] & b\alpha \arrow[u, "b\alpha"'] & & \langle y\rangle\omega_0 \arrow[u, dotted, "(\mathsf{Id}_A\ a\ b)\alpha\uparrow_x\langle y\rangle\omega_0"']
\end{tikzcd}$$

We give the interpretation of $\Gamma \vdash \mathsf{Ref}\,a : \mathsf{Id}_A\ a\ a$ given $\Gamma \vdash a : A$. For any set of directions I, and any I-cube ρ, we have to give a line $a\rho \to a\rho$. For this, we choose a direction x not in I and we define $(\mathsf{Ref}\,a)\rho = \langle x\rangle a\rho\iota_x$, which can also simply be written $(\mathsf{Ref}\,a)\rho = \langle x\rangle a\rho$.

Next we show that we can interpret an elimination operator for the identity type. Suppose $\Gamma \vdash a : A$, $\Gamma \vdash b : A$, $\Gamma \vdash u : \mathsf{Id}_A\ a\ b$ and $\Gamma.A \vdash P$ and $\Gamma \vdash v : P[a]$. We will define an operator

$$\Gamma \vdash \mathsf{T}(u, v) : P[b].$$

Let ρ be some I-cube of Γ. Then $u\rho$ is of the form $\langle x\rangle\omega$ for some path $\omega : a\rho \to_x b\rho$, x not in I, $\omega \in A\rho$. The I, x-cube (ρ, ω) in $\Gamma.A$ is then a path $[a]\rho \to_x [b]\rho$ and we define (see the picture below)

$$\mathsf{T}(u, v)\rho = P(\rho, \omega)^+ v\rho \quad \text{where} \quad \langle x\rangle\omega = u\rho$$

The condition $(\mathsf{T}(u, v)\rho)f = \mathsf{T}(u, v)(\rho f)$ follows from the uniformity condition on the Kan filling operations.

$$\begin{tikzcd}
b\rho & {[b]\rho} & \mathsf{T}(u,v)\rho \\
a\rho \arrow[u, "\omega"] & {[a]\rho} \arrow[u, "{\rho,\omega}"] & v\rho \arrow[u, dotted, "P(\rho,\omega)\uparrow v\rho"]
\end{tikzcd}$$

We have that $P(\rho, \omega){\uparrow}v\rho$ is a line connecting $v\rho$ and $\mathsf{T}(u, v)\rho$. In particular for $u = \mathsf{Ref}\,a$, this gives an interpretation of an operator

$$\Gamma \vdash \mathsf{H}(v) : \mathsf{Id}_{P[a]}\ v\ \ \mathsf{T}(\mathsf{Ref}\,a, v)$$

by taking $\mathsf{H}(v)\rho = \langle x\rangle P(\rho\iota_x, a\rho){\uparrow}v\rho$. The computation rule for identity is thus only validated by a path to v via $\mathsf{H}(v)$[3].

[3] The validity of the computation rule for identity corresponds to considering only fibrations that are *regular* in the sense of Hurewicz [14].

$$\frac{\Gamma \vdash A \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash \mathsf{Id}_A\ a\ b} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{Ref}\,a : \mathsf{Id}_A\ a\ a}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : A \quad \Gamma \vdash u : \mathsf{Id}_A\ a\ b \quad \Gamma.A \vdash P \quad \Gamma \vdash v : P[a]}{\Gamma \vdash \mathsf{T}(u, v) : P[b]}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash P \quad \Gamma \vdash v : P[a]}{\Gamma \vdash \mathsf{H}(v) : \mathsf{Id}_{P[a]}\ v\ \ \mathsf{T}(\mathsf{Ref}\,a, v)}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{center}\,(a, \mathsf{Ref}\,a) : \Pi\, T(\mathsf{Id}_{T\mathsf{p}}(a, \mathsf{Ref}\,a)\ \mathsf{q})} \quad \text{where } T = \Sigma\, A\,(\mathsf{Id}_{A\mathsf{p}} a\mathsf{p}\ \mathsf{q})$$

Figure 2 Rules for Id-types.

We finally show that, given $\Gamma \vdash a : A$, the type $\Gamma \vdash T = \Sigma\ A\ (\mathsf{Id}_{A\mathsf{p}}\ a\mathsf{p}\ \mathsf{q})$ is contractible. For this we have to find a center of T and a path to this center for any element of T. That is, we have to find two sections $\Gamma \vdash t : T$ and $\Gamma.T \vdash u : \mathsf{Id}_{T\mathsf{p}}\ t\mathsf{p}\ \mathsf{q}$. We define $t = (a, \mathsf{Ref}\,a)$. Let ρ be some I-cube of Γ and let $(v, \langle x\rangle\alpha)$ be some element of $T\rho$. So v is an element of $A\rho$ and α is a line connecting $a\rho$ and v in some direction x not in I. We introduce a direction y not in I, x and define:

$$u(\rho, (v, \langle x\rangle\alpha)) = \langle y\rangle(A\rho^{+}_{x,y}(a\rho, a\rho, \alpha), \langle x\rangle A\rho{\uparrow}_{x,y}(a\rho, a\rho, \alpha))$$

The fact that the filling operations commute with substitution ensures that this defines a section $\Gamma.T \vdash u : \mathsf{Id}_{T\mathsf{p}}\ t\mathsf{p}\ \mathsf{q}$.

We summarize the rules we interpret in the Kan cubical set model in Figure 2, where we left out the equations that the operations commute with substitutions, e.g. $(\mathsf{Id}_A\ a\ b)\sigma = \mathsf{Id}_{A\sigma}\ a\sigma\ b\sigma$.

N.A. Danielsson has checked formally in Agda that these properties are enough to develop all basic propositions of univalent mathematics; this Agda development[4] is accompanying the paper [6].

Let us define the more common elimination operator of C. Paulin-Mohring [18] from the above operations—with the difference that its computation rule only holds propositionally, and not as usual definitionally. In order not to make the notation too heavy we'll use informal reasoning in type theory; note that the definition can be given internally in type theory and we don't refer to the model; this definition follows N.A. Danielsson's Agda development (loc. cit.). First note that using the transport operation T one can define composition $p \circ q : \mathsf{Id}_A a\ c$ of two identity proofs $p : \mathsf{Id}_A a\ b$, $q : \mathsf{Id}_A b\ c$, as well as inverses $p^{-1} : \mathsf{Id}_A b\ a$. With H one can derive $\mathsf{Id}_{\mathsf{Id}_A a\ a}(p^{-1} \circ p)\ (\mathsf{Ref}\,a)$.

Let A be a type, $a : A$, and $C(b, p)$ a type given $b : A$, $p : \mathsf{Id}_A a\ b$, such that $v : C(a, \mathsf{Ref}\,a)$; for $b : A$ and $p : \mathsf{Id}_A a\ b$ we define $\mathsf{J}(a, v, b, p) : C(b, u)$. We can consider C as a dependent type over $T = (\Sigma x : A)\mathsf{Id}_A a\ x$ via $C(w.1, w.2)$ for $w : T$. As we showed in the last paragraph, T is contractible with center $(a, \mathsf{Ref}\,a)$, and thus we get a witness $\mathsf{app}(h, (b, p)) : \mathsf{Id}_T(a, \mathsf{Ref}\,a)\ (b, p)$ for $h = \lambda u$, u as in the above paragraph; now with T (w.r.t. the type $C(w.1, w.2)$ for $w : T$) we can define

$$\mathsf{J}(a, v, b, p) = \mathsf{T}(\mathsf{app}(h, (a, \mathsf{Ref}\,a))^{-1} \circ \mathsf{app}(h, (b, p)), v).$$

[4] Available at: `http://www.cse.chalmers.se/~nad/`

Now if $p = \mathsf{Ref}\,a$, we get that $\mathsf{app}(h,(a,\mathsf{Ref}\,a))^{-1} \circ \mathsf{app}(h,(b,p))$ is propositionally equal to $\mathsf{Ref}(\mathsf{Ref}\,a)$, and thus using T and H again one gets a witness of $\mathsf{Id}_{C(a,\mathsf{Ref}\,a)}\ v\ \ \mathsf{J}(a,v,a,\mathsf{Ref}\,a)$.

Even though J doesn't satisfy the judgmental equality, the model validates a new operation $\mathsf{mapOnPaths}$ which behaves well w.r.t. judgmental equality. Its rule given $\Gamma \vdash A$, $\Gamma \vdash B$, $\Gamma \vdash u : A$ and $\Gamma \vdash v : A$ is

$$\frac{\Gamma \vdash \varphi : A \to B \quad \Gamma \vdash p : \mathsf{Id}_A\ u\ v}{\Gamma \vdash \mathsf{mapOnPaths}(\varphi, p) : \mathsf{Id}_B\ (\mathsf{app}(\varphi, u))\ (\mathsf{app}(\varphi, v))}$$

where $A \to B$ is the non-dependent function space $\Pi A(B\mathsf{p})$. Given ρ in $\Gamma(I)$ we define $\mathsf{mapOnPaths}(\varphi,p)\rho = \langle x\rangle\ (\varphi\rho)_1\omega$ for $p\rho = \langle x\rangle\omega$. This satisfies the equations

$$\begin{aligned}
\mathsf{mapOnPaths}(\mathrm{id}, p) &= p\\
\mathsf{mapOnPaths}(\varphi \circ \psi, p) &= \mathsf{mapOnPaths}\big(\varphi, \mathsf{mapOnPaths}(\psi, p)\big)\\
\mathsf{mapOnPaths}(\varphi, \mathsf{Ref}\,a) &= \mathsf{Ref}(\mathsf{app}(\varphi, a))\\
\mathsf{mapOnPaths}(\lambda(b\mathsf{p}), p) &= \mathsf{Ref}\,b
\end{aligned}$$

where now $\varphi \circ \psi$ denotes ordinary function composition and $\lambda(b\mathsf{p})$ is constant.

Notice that some of these equations do *not* hold if the identity type is defined as an inductive family, as in [17].

This interpretation of identity satisfies function extensionality (left to the reader).

8.3 Description of a universe

We now describe the interpretation of U as a universe of Kan cubical sets. We give U only as a cubical set (following [12, 23]) and only indicate how an operation similar to the Kan fillings can be given. The full proof that U has a Kan structure will be presented in the forthcoming [13].

Recall that the Yoneda embedding is denoted by $\mathbf{y}$. An element A of $U(I)$ is a type $\mathbf{y}I \vdash A$ with Kan structure such that for each $f : I \to J$ the set A_f is small (we use subscripts to keep the notation separate from the restrictions). Given such a $\mathbf{y}I \vdash A$ and $f : I \to J$ the restriction Af of A by f is defined to be $\mathbf{y}J \vdash A(\mathbf{y}f)$, where $\mathbf{y}f : \mathbf{y}J \to \mathbf{y}I$ is the substitution induced by f; thus $(Af)_g = A_{fg}$. This defines U as a cubical set.

Note that the points of U are simply the (small) uniform Kan cubical sets. More precisely, since $\emptyset$ is initial in $\mathcal{C}$, any A in $U(\emptyset)$ becomes a cubical set when we define $A(I)$ as A_f for the unique $f : \emptyset \to I$. A line in U between points A and B can be seen as a "heterogeneous" notion of lines, cubes, ... $a \to b$ where a is an I-cube of A and b an I-cube of B.

As a first step towards proving that this cubical set satisfies the Kan condition we show how to compose an A and B in $U(I)$ with $x \in I$ assuming $A(x = 1) = B(x = 0)$; we define $C = \mathsf{comp}(A,B) \in U(I)$ such that $C(x = 0) = A(x = 0)$, $C(x = 1) = B(x = 1)$, and for $f : I \to J$ defined on x, $Cf = \mathsf{comp}(Af, Bf)$. (Compare this to the composition of relations.)

We define the sets C_f, $f : I \to J$ by case distinction on $f(x)$; in case $f(x) = 0$, we can write $f = (x = 0)f'$ and we have to set $C_f = A_f$ as we have to satisfy $C_f = (C(x = 0))_{f'} = (A(x = 0))_{f'} = A_f$; similarly, if $f(x) = 1$, we set $C_f = B_f$. In case, f is defined on x, an element of C_f is any pair (a,b) such that $a \in A_f$ and $b \in B_f$ with $a(x = 1) = b(x = 0)$ in $A_{f(x=1)} = A(x = 1)_{(f-x)} = B(x = 0)_{(f-x)} = B_{f(x=0)}$.

We still have to define the restrictions $C_f \to C_{fg}$ for $g : J \to K$; in the first two cases from above, the restrictions are induced by A_f and B_f respectively. In case f is defined on x, we look at $g(f(x))$: if $g(f(x)) = 0$, we set $(a,b)g = ag$; if $g(f(x)) = 1$, we set $(a,b)g = bg$; and if g is defined at $f(x)$, we define $(a,b)g = (ag, bg)$.

It remains to define the Kan fillings for C; it suffices to give them for C_1 as C_f is either determined by A_f, B_f, or $\mathsf{comp}(Af, Bf)_1$; so let $J, x' \subseteq I$ with $x' \notin J$, and $\vec{u}$ be a open box in C_1, i.e. $u_{yc} \in C_{(y=c)}$ for $(y, c) \in O^+(J, x')$ with $u_{yc}(z = d) = u_{zd}(y = c)$. Note that for $y \neq x$, $u_{yc} = (a_{yc}, b_{yc})$ with $a_{yc} \in A_1$ and $b_{yc} \in B_1$ with $a_{yc}(x = 1) = b_{yc}(x = 0)$. We want to define $u = C_1{\uparrow}\vec{u}$. There are three cases. First, in case $x = x'$, we set $a_{x0} = u_{x0} \in C_{(x=0)} = A_{(x=0)}$; this yields an open box $\vec{a}$ in A_1 which we can fill to $a = A_1{\uparrow}\vec{a} \in A_1$. Now setting $b_{x0} = a(x = 1)$ yields an open box $\vec{b}$ in B_1 which we can fill to get $b = B_1{\uparrow}\vec{b} \in B_1$. Note that $b(x = 0) = a(x = 1)$ and thus we can set $u = (a, b)$.

Second, in case $x \neq x'$ with $x \in J$, we construct an element $v \in A_{(x=1)} = B_{(x=0)}$ first. For $(y, c) \in O^+(J - x, x')$ define $v_{yc} = a_{yc}(x = 1)$ (which is also equal to $b_{yc}(x = 0)$). It is readily checked that this defines an open box in $A_{(x=1)} = B_{(x=0)}$ and thus we get $v = A_{(x=1)}{\uparrow}\vec{v}$. Now set $a_{x1} = b_{x0} = v$; this yields open boxes $\vec{a}$ and $\vec{b}$ in A_1 and B_1, respectively. Thus we can take $u = (A_1{\uparrow}\vec{a}, B_1{\uparrow}\vec{b})$.

Finally, in case $x \notin J$, we directly have open boxes $\vec{a}$ and $\vec{b}$ in A_1 and B_1, respectively. Setting $u = (A_1{\uparrow}\vec{a}, B_1{\uparrow}\vec{b})$ gives an element in C_1 since

$$(A_1{\uparrow}\vec{a})(x = 1) = A_{(x=1)}{\uparrow}(\vec{a}(x = 1)) = B_{(x=0)}{\uparrow}(\vec{b}(x = 0)) = (B_1{\uparrow}\vec{b})(x = 0).$$

This concludes the definition of $C = \mathsf{comp}(A, B)$.

8.4 Equivalence and equality of types

We explain in this section how to transform any equivalence $\sigma\colon A \to B$ between two small Kan cubical sets to a path $A \to B$ in U, as defined in the previous section. Let us recall the notion of equivalence between types (cf. [24, Definition 4.4.1]) using informal notation. For a type A we define the proposition of being contractible $\mathsf{isContr}\, A$ to be $(\Sigma a : A)(\Pi x : A)\, \mathsf{Id}_A\ a\ x$. The *fiber* $\mathsf{fib}_\sigma\, b$ of a map $\sigma\colon A \to B$ over $b : B$ is defined as $(\Sigma x : A)\, \mathsf{Id}_B\ \mathsf{app}(\sigma, x)\ b$. A map $\sigma\colon A \to B$ is an *equivalence* if all its fibers are contractible, i.e. if

$$(\Pi b : B)\ \mathsf{isContr}(\mathsf{fib}_\sigma\, b).$$

This amounts to give $\varphi : (\Pi b : B)(\Sigma x : A)\, \mathsf{Id}_B\ \mathsf{app}(\sigma, x)\ b$ and $\psi : (\Pi b : A)(\Pi u : \mathsf{fib}_\sigma\, b)\, \mathsf{Id}_{\mathsf{fib}_\sigma\, b}\ \mathsf{app}(\varphi, b)\ u$. If we now assume that A and B are Kan cubical sets (which corresponds to types in the empty context), this definition unfolds to the following data: a map $\sigma : A \to B$ is an equivalence if there is a map $\delta : B \to A$ and a map assigning to b a line $b'\colon \sigma\delta b \to b$, and a transformation of any equality $\omega : \sigma a \to b$, where a (resp. b) is an I-cube of A (resp. B) to a "square" (really a pair of an I, x-cube of A and an I, x, y-cube of B)

$$\begin{tikzcd} a \arrow[r, "\omega^*"] & \delta b \end{tikzcd}$$

$$\begin{tikzcd} \sigma a \arrow[r, "\sigma\omega^*"] \arrow[d, "\omega"'] & \sigma\delta b \arrow[d, "b'"] \\ b \arrow[r, "b"'] & b \end{tikzcd}$$

We define from this a path C between A and B in the direction x. For any substitution $f : \{x\} \to I$ we have to define a set C_f together with substitution maps $C_f \to C_{fg}$. If

$f(x) = 0$ we take $C_f = A(I)$ and if $f(x) = 1$ we take $C_f = B(I)$. If $f(x) = y$ then we define C_f to be the set of pairs (a, b) where a is an $(I - y)$-cube of A and b is an I-cube of B and $b(y = 0) = \sigma a$. It can be then be checked in an elementary way that if σ is an equivalence, then this "heterogeneous" notion of cube has the uniform Kan property.

In pictures, the main difficult case is to complete an open box

$$\begin{tikzcd}
\sigma a_0 \arrow[r] & b_0 \arrow[d] \\
\sigma a_1 \arrow[r] & b_1
\end{tikzcd}$$

to a square

$$\begin{tikzcd}
a_0 \arrow[d] & \sigma a_0 \arrow[r] \arrow[d] & b_0 \arrow[d] \\
a_1 & \sigma a_1 \arrow[r] & b_1
\end{tikzcd}$$

For this, using the fact that σ is an equivalence, we transform the open box in an open box in A

$$\begin{tikzcd}
a_0 \arrow[r] & \delta b_0 \arrow[d] \\
a_1 \arrow[r] & \delta b_1
\end{tikzcd}$$

and since A is Kan, it can be filled to a box

$$\begin{tikzcd}
a_0 \arrow[r] \arrow[d] & \delta b_0 \arrow[d] \\
a_1 \arrow[r] & \delta b_1
\end{tikzcd}$$

and we can then fill the box in B

$$\begin{tikzcd}
a_0 \arrow[ddd] & \sigma a_0 \arrow[rrr] \arrow[ddd] \arrow[dr] & & & b_0 \arrow[ddd] \arrow[dl] \\
& & \sigma\delta b_0 \arrow[r] \arrow[d] & b_0 \arrow[d] & \\
& & \sigma\delta b_1 \arrow[r] & b_1 & \\
a_1 & \sigma a_1 \arrow[rrr] \arrow[ur] & & & b_1 \arrow[ul]
\end{tikzcd}$$

Since our model is constructive, this gives a way to effectively transport properties and structures on a Kan cubical set to one which is equivalent. In particular we can effectively transport properties and structures of a groupoid to one which is categorically equivalent.

We have only described here a weak corollary of the Axiom of Univalence, but the complete Axiom can be validated in this model as well[5] and will be presented in a forthcoming publication.

8.5 Propositional reflection

We can describe the operation of Kan "completion". Given a cubical set X we add operations X^+, $X{\uparrow}$, X^-, $X{\downarrow}$ in a *free* way, i.e. considering these operations as *constructors*. At the same time one defines the restrictions of the added operations, resulting in an inductive-recursive definition. The uniformity condition determine what the restrictions of these elements should. In this way we get a new cubical set Y, satisfying by definition the Kan extension property, with a map $X \to Y$. Furthermore, if Z is Kan, and we have a map $\sigma : X \to Z$ there is a map $Y \to Z$ extending σ. This map is furthermore unique if we impose it to commute with the Kan operations. In general however, the maps of Kan cubical sets do not need to commute with the Kan operations.

The same idea can be used to define $\mathsf{inh}\, X$, the *proposition* stating that X is inhabited. Besides adding constructors $(\mathsf{inh}\, X)^+$, $(\mathsf{inh}\, X){\uparrow}$, $(\mathsf{inh}\, X)^-$ and $(\mathsf{inh}\, X){\downarrow}$, we also add a constructor $\alpha_x(u_0, u_1)$ connecting formally along the dimension x any two I-cubes u_0 and u_1 (with x not in I) and constructors for the Kan filling and composition operations. Thus each I-cube u in $\mathsf{inh}\, X$ is of one of the forms: either u an I-cube of X; a formal Kan filling, e.g. $(\mathsf{inh}\, X){\uparrow}\vec{u}$ with $\vec{u}$ an open box in $\mathsf{inh}\, X$; or of the form $\alpha_x(u_0, u_1)$ with u_i in $(\mathsf{inh}\, X)(I - x)$. At the same time we define the restrictions

$$\alpha_x(u_0, u_1)(x = 0) = u_0 \qquad \alpha_x(u_0, u_1)(x = 1) = u_1$$

and, if f is defined on x with $y = f(x)$,

$$\alpha_x(u_0, u_1)f = \alpha_y(u_0(f - x), u_1(f - x)).$$

This satisfies the required induction principle of $\mathsf{inh}\, X$: if we have a map $\varphi : X \to Y$, we can extend this to a map $\tilde{\varphi} : \mathsf{prop}\, Y \times \mathsf{inh}\, X \to Y$ where $\mathsf{prop}\, Y$ is $(\Pi y_0\ y_1 : Y)\, \mathsf{Id}_Y\ y_0\ y_1$. For $p \in (\mathsf{prop}\, Y)(I)$ and $u \in (\mathsf{inh}\, X)(I)$ we define $\tilde{\varphi}(p, u)$ in $Y(I)$ by induction on u. The difficult case is when u is $\alpha_x(u_0, u_1)$ with $x \in I$ and $u_i \in (\mathsf{inh}\, X)(I - x)$. By induction hypothesis, we already defined $v_i = \tilde{\varphi}(p(x = i), u_i) \in Y(I - x)$. Applying $p(x = 0)$ to both v_0 and v_1 gives a path $\langle x\rangle\omega$, where $\omega \in Y(I)$ connecting v_0 to v_1 along x, and we set $\tilde{\varphi}(p, u) = \omega$. Note that the choice of $p(x = 0) \in (\mathsf{prop}\, Y)(I - x)$ above is *not* canonical.

We can also define the spheres. For instance S^1 will be the Kan completion of the cubical set generated by a point base and a loop loop.

We can then define $\exists\ A\ B$ to be $\mathsf{inh}(\Sigma\ A\ B)$. If $\Sigma\ A\ B$ is a proposition we have an inhabitant of $\exists\ A\ B \to \Sigma\ A\ B$ and this can be seen as a generalization of the *axiom of description* since if A set, B proposition and B is satisfied by at most one element of A then $\Sigma\ A\ B$ is a proposition.

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[5] The algorithms can be found in the implementation available at `http://github.com/simhu/cubical`.

Acknowledgments. The research for this paper has been started while the first two authors were members of the Institute for Advanced Study in Princeton, as part of the program *Univalent Foundations of Mathematics*. We are grateful for the generous support by the IAS and the Fund for Math.

The last two authors acknowledge financial support from the ERC: The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement nr. 247219.

The authors wish to thank Jean-Philippe Bernardy, Cyril Cohen, Andy Pitts and Michael Shulman for stimulating discussions on the topic of this paper. The clear presentation of [27] provided an important help.

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